

Low- ℓ power suppression in punctuated inflation

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Abstract. Motivated by Planck confirmation of an anomalously low value of the CMB temperature fluctuations up to multipole $\ell < 40$, we in this paper try to explain such feature by investigating case of punctuated inflation scenario. This form of inflation potential is inspired by MSSM wherein suppression of curvature perturbation power at large scales is produced by introducing period of fast-roll phase of the inflation sandwiched between two stages of slow-roll phase.

We apply Markov Chain Monte Carlo analysis to determine posterior distribution and the best fit values of the model parameters using recent WMAP9 and Planck data. We show that WMAP9 and Planck results are consistent with each other. We shown that the Planck data gives much tighter constraints for punctuated inflation parameters. We find that punctuated inflation leads to better fit in CMB data compared to simple power law model. For WMAP9 data the improvement in fit is marginal ($\Delta\chi^2 \sim 4$), however, for WMAP9+Planck the improvement is significant ($\Delta\chi^2 \sim 17$). We find that *AIC* does not discriminate between punctuated inflation and simple power law model for WMAP9 data. However, for WMAP9+Planck data we find that punctuated inflation is strongly favored over a simple power law spectrum.

Keywords: Cosmology, CMB

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1 Introduction

Anisotropy in the cosmic microwave background (CMB) is one of the most robust and informative tools in cosmology. Measurements of the CMB anisotropies by COBE, WMAP9 [1, 2] and Planck [3, 4] are in striking agreement with Λ CDM cosmology. These results are consistent with an adiabatic and nearly scale invariant spectrum of primordial perturbations, as predicted by the simple inflationary models [5–7]. The inflationary model(s) provide us potential solutions to most of the outstanding problems of standard Big Bang. They successfully explain the flatness problem, horizon problem and the monopole problem [5, 6, 8]. A key feature of these models is that during inflation epoch, in the very early universe, primordial quantum fluctuations [9–11] of some scalar fields are produced which eventually transform into large scale structure of the universe under the gravitational instability paradigm. The observed statistical distribution of temperature fluctuations in the cosmic microwave background is believed to be sourced by these quantum fluctuations.

Although, power law (PL) model with almost scale-invariant, adiabatic and Gaussian primordial spectrum “ $\mathcal{P}(k)$ ” (k being the co-moving wavenumber) has emerged as the most successful model from the recent observations, there exist several anomalies in CMB data which could have far reaching consequences. Some of these anomalies are low CMB power at large angular scales [12–15], departure from either Gaussianity or statistical isotropy [16–21], lack of correlations on large scales [22–24], hemispheric asymmetry and the cold spot [4, 25–27], the so-called Axis of Evil [28–30] and oscillatory features in the primordial power spectrum [31–39]. Many inconsistencies in angular power spectrum have also been found between WMAP9 and Planck data [40–42]. Recently, Addison et al. [43] found internal inconsistencies between high ($\ell > 1000$) and low ($\ell < 1000$) multipoles in the Planck data. They suggest that tension in the Planck high ℓ spectrum is either due to an unlikely statistical

fluctuation or unaccounted systematics in data. Similarly, Hazra & Shafieloo [44] found that Planck data is consistent to the concordance Λ CDM model only at $2\text{--}3\sigma$ confidence level with an indication of lack of power at both high and low ℓ 's with respect to concordance model. The study of these anomalies and their statistical significance is of utmost importance for cosmology as these results could test the Λ CDM model.

Lack of power at low CMB multipoles can be attributed to spatial curvature [45], non-trivial topology [46], violation of statistical anisotropies [16], hemispherical anisotropy and non-gaussianity [47, 48], bouncing inflation [49, 50], dark energy during the inflation [51], primordial micro black holes [52], loop quantum cosmology [53], string theory [54] and so on. Besides, there exist many other classes of inflationary models which give rise to additional features in primordial power spectra compared to simple power law model. Exploring different inflationary models is still an interesting problem and could help us in explaining many anomalies. In our earlier work, we carried a detailed study of the inflationary models which could produce cut off in primordial power spectrum at low k so as to reproduce necessary suppression in the CMB power spectrum up to multipoles $\ell \leq 40$ observed in the Planck data. We found marginal preference of the cut off parameters that describe the power suppression at low multipoles [15]. However, in all these models, we have to assume a specific pre-inflationary era (like kinetic or radiation dominated regime) and/or some special initial conditions to achieve a near scale invariant power spectrum along with a cut off at large scales (i.e low k). In this paper, we will consider punctuated inflationary scenario (PI model) which allows a brief period of fast roll sandwiched between two stages of slow roll inflation. Such scenario can be encountered in the Minimal Supersymmetric Standard Model (MSSM) wherein the inflationary potentials contain a point of inflection [55–59]. The break in the slow roll will induce power suppression followed by rise in the primordial power spectrum corresponding to the scales that leave the Hubble radius just before the transition to the fast roll [60–62]. The form of inflationary potential that we consider in this work has been previously investigated in [62, 63] using WMAP9 5-year data and this work can be regarded as a progression of their work. They showed that brief period of fast roll induces step like feature in the primordial spectrum corresponding to the Hubble scales leading to the suppression in the CMB power at low multipoles. Such scenario is advantageous in that we don't have to consider any special pre-inflationary era or impose any special initial condition. Given the new Planck data and improved WMAP9 data, we aim to tighten the constraints on the parameter space with respect to previous work. Indeed, if the Planck and WMAP9 results are not consistent with each other then this would clearly imply the rejection of such model. Moreover, we also use Akaike information criterion (*AIC*) [64] to compare punctuated inflation model with PL model.

The plan of this paper is as follows: In section 2, we discuss punctuated inflationary model which can produce cut off primordial power spectrum at large scales. We then discuss CMB data set and methodology used in our analysis in section 3. In section 4, we give parameter estimates and explore Akaike information criterion for comparing punctuated inflationary model with power law model. Finally, we conclude our work in section 5. Throughout this work we will assume $\hbar = c = 8\pi G = 1$.

2 Punctuated inflation

The deviations from the scale invariance of the power spectrum can be produced by introducing two or more periods of fast roll phase. Such cases are not uncommon in double

inflationary scenarios where one uses two scalar fields [65–68]. Within single field inflationary scenarios, such departure from the slow roll phase are usually produced by introducing step or sudden change in slope of the inflationary field [69–71]. However, there also exist many smooth and well behaved potentials that can incorporate fast roll phase [60, 61, 72]. The extent of deviations would depend on the nature and duration of the fast roll which are essentially controlled by the model parameters. The potential that we consider in this work to achieve a phase of fast roll is motivated by Minimal Super-symmetric Standard Model (MSSM), an extension of the Standard Model (SM) which has many cosmological consequences [55–59]. In such scenarios, inflation carries the standard model (SM) charges and eventually decays into the SM baryons and the cold dark matter. We will consider following form of inflationary potential which has been earlier studied by Jain et al. [62, 63]

$$V(\phi) = \left(\frac{m^2}{2}\right) \phi^2 - \left(\frac{\sqrt{2\lambda(n-1)m}}{n}\right) \phi^n + \left(\frac{\lambda}{4}\right) \phi^{2(n-1)}, \quad (2.1)$$

where we consider the case $n = 3$ and the coefficient of the ϕ^n term has been chosen in such a way that the potential has an point of inflection (point where both $V_\phi \equiv (dV/d\phi)$ and $V_{\phi\phi} \equiv (d^2V/d\phi^2)$ vanish) at $\phi = \phi_0$ given by

$$\phi_0 = \left(\frac{2m^2}{(n-1)\lambda}\right)^{\frac{1}{2(n-2)}}. \quad (2.2)$$

This form of the inflationary potential is shown in figure 1 using best fit parameters. It is important to note that in the usual Minimal Super-symmetric Standard Models, the point of the inflection is located at the sub-Planckian values (i.e. $\phi_0 \ll 1$). However, for our case the inflationary potential has a inflection at $\phi_0 \gg 1$, if we have to induce a period of the fast roll followed by a second phase of slow roll. This is helpful in a way that in our case the inflationary phase becomes independent of the initial value of the ϕ_i and $\dot{\phi}_i$ while in case of MSSM they have to be severely fine tuned; $\phi_i \sim \phi_0$ and $\dot{\phi}_i \sim 0$. It has been shown in [62] that if we start with any initial value $\phi_i \gg \phi_0$ then the subsequent dynamics of the field approach the attractor and the attractor trajectory exhibits two regimes of slow roll inflation sandwiching a period of fast roll. It is worth mentioning that even though we consider the parameters of the inflationary potential that are different from the MSSM case, it may be possible to realize such scenarios beyond standard model (see [62] for more details).

It is often found that if we consider single field inflationary models along with the slow roll approximation then the amplitude of the scalar perturbations freezes after the modes leave the Hubble radius which can be expressed in terms of inflationary potential as follows [7, 73]

$$\mathcal{P}_S(k) \simeq \left(\frac{1}{12\pi^2}\right) \left(\frac{V^3}{V_\phi^2}\right). \quad (2.3)$$

However, if we also consider an intermediate period of fast roll, it is found that the super-Hubble amplitude of the modes that exit the Hubble scale just before start of fast roll are enhanced compared to their value at Hubble exit. For modes that exit the Hubble scale during fast roll, the amplitudes are actually suppressed [14, 60, 61, 74, 75]. Further, the modes that leave well before and after fast roll remain unaffected.

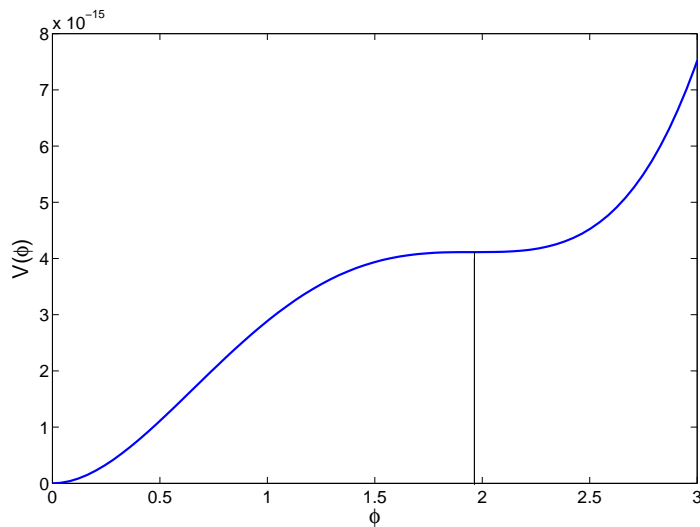


Figure 1. Form of the inflationary potential for parameters: $m = 1.1323 \times 10^{-7}$ and $\lambda = 3.3299 \times 10^{-15}$ (or $\phi_0 = 1.9622$) which turn out to be the best fit values for the WMAP9+Planck data set. The vertical line represents the inflection point.

3 Solving primordial power spectrum

In this section, we will discuss equations of motion governing the evolution of scalar (curvature) perturbations and tensor perturbations. We use the formalism similar to Aich et al. [35], Hazra et al. [62] and Jain et al. [76] where they evolve the perturbations using the exact numerical approach using e-folds as the independent variable.

3.1 Background equations

The general action of scalar field in the curved space-time can be described by [77, 78]

$$S = - \int d^4x \sqrt{|g|} [\mathcal{L}_g + \mathcal{L}_\phi], \quad (3.1)$$

where

$$\mathcal{L}_g = \frac{1}{2}R, \quad \mathcal{L}_\phi = \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi), \quad (3.2)$$

are the Lagrangians for the gravitation and scalar fields respectively, $V(\phi)$ is the potential energy of the scalar field and R is the curvature scalar. Variation of the action with respect to ϕ leads to the Klein-Gordon equation [78]

$$\frac{1}{\sqrt{|g|}}\partial_\mu \left[\sqrt{|g|}g^{\mu\nu}\partial_\nu\phi \right] + V_\phi = 0. \quad (3.3)$$

For the spatially flat FRW metric where $g_{\mu\nu} = \text{diag}(1, -a^2, -a^2, -a^2)$ and $\sqrt{|g|} = a^3$, a being the scale factor, the above equation reduces to [7, 8]

$$\ddot{\phi} + 3H\dot{\phi} - a^{-2}\nabla^2\phi + V_\phi = 0. \quad (3.4)$$

The second term provides friction term for the scalar field which is proportional to the Hubble constant (expansion rate) $H = \dot{a}/a$. Third term in the above equation approximately equal

to zero as the inflation would rapidly smooth out spatial variations in the universe. Similarly, variation of the action with respect to $g^{\mu\nu}$ gives us the Einstein tensor $G_{\mu\nu}$

$$G_{\mu\nu} = T_{\mu\nu}, \quad (3.5)$$

where $T_{\mu\nu}$ is energy-momentum tensor of the scalar field given by

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \mathcal{L}_\phi. \quad (3.6)$$

For spatially flat FRW metric $T_{\mu\nu}$ reduces to [78]

$$T_{00} = \rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \quad T_{ii} = p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi), \quad (3.7)$$

where ρ_ϕ and p_ϕ are the energy density and pressure density of the scalar field. The combination of G_{00} and G_{ii} provide us a useful equation

$$\dot{H} = \frac{1}{2}\dot{\phi}^2, \quad 3H^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi), \quad (3.8)$$

where the second equation is called the Friedmann equation. For inflation to happen, the scalar field is assumed to vary slowly with time (i.e $\dot{\phi}/2 \ll V$) which leads to the exponentially growing scale factor in the Friedmann equation. Slow-roll inflation is characterized by two parameters $\epsilon \ll 1$ and $\eta \ll 1$ which are defined as [6]

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{1}{2} \left(\frac{V_\phi}{V(\phi)} \right)^2, \quad \eta = \epsilon + \delta = \left(\frac{V_{\phi\phi}}{V(\phi)} \right), \quad (3.9)$$

where $\delta = -\ddot{\phi}/H\dot{\phi}$.

3.2 Perturbation equations

In spatially flat Friedmann universe, the Fourier modes of the curvature perturbation \mathcal{R} and the tensor perturbation h are described by the following equations [73]

$$\mathcal{R}_k'' + 2\frac{z'}{z}\mathcal{R}_k' + k^2\mathcal{R}_k = 0, \quad h_k'' + 2\frac{a'}{a}h_k' + k^2h_k = 0, \quad (3.10)$$

where $z = a\dot{\phi}/H$ and the primes represent differentiation with respect to the conformal time. It is advantageous to consider the gauge invariant Mukhanov variables u and v [73]

$$u = z\mathcal{R}, \quad v = ah. \quad (3.11)$$

The Fourier components u_k and v_k obey the following equation of motions

$$\ddot{u}_k + \left(k^2 - \frac{\ddot{z}}{z} \right) u_k = 0, \quad \ddot{v}_k + \left(k^2 - \frac{\ddot{a}}{a} \right) v_k = 0, \quad (3.12)$$

We can then define the primordial power spectrum of curvature perturbations $\mathcal{P}_s(k)$ and tensor perturbations $\mathcal{P}_t(k)$ by the two-point correlation functions as [79]

$$\langle \mathcal{R}_{k_1} \mathcal{R}_{k_2}^* \rangle = \frac{2\pi^2}{k^3} \mathcal{P}_s(k) \delta^{(3)}(k_1 - k_2), \quad \langle h_{k_1} h_{k_2}^* \rangle = \frac{2\pi^2}{k^3} \mathcal{P}_t(k) \delta^{(3)}(k_1 - k_2). \quad (3.13)$$

Assuming Gaussianity and adiabaticity, $\mathcal{P}_s(k)$ and $\mathcal{P}_t(k)$ encode all the information for a complete statistical description of the fluctuations and are related to u_k and v_k via

$$\mathcal{P}_s(k) = \frac{k^3}{2\pi^2} \left| \frac{u_k}{z} \right|^2, \quad \mathcal{P}_t(k) = 2 \frac{k^2}{2\pi^2} \left| \frac{v_k}{a} \right|^2 \quad (3.14)$$

Since, tensor-to-scalar ratio “ r ” remains smaller than 10^{-4} over whole cosmological scale for our model, therefore, we will only consider scalar perturbation to minimize the computation time.

3.3 Initial conditions

In order to evaluate scalar power spectrum for different modes of k , we need to solve together background equation (equation 3.4) and perturbation equation (equation 3.10). Since both the equations are second order differential equations, therefore, we need 4 initial conditions - ϕ_i , $\dot{\phi}_i$, \mathcal{R}_i and \mathcal{R}'_i . We assume ϕ_i to be equal to 12 and $\dot{\phi}_i$ is evaluated using

$$3H\dot{\phi}_i = -V(\phi_i). \quad (3.15)$$

The initial conditions for \mathcal{R}_i and \mathcal{R}'_i can be obtained by imposing Bunch-Davies vacuum of de Sitter space when the perturbations are well inside the origin

$$\mathcal{R}_i = \frac{1}{\sqrt{2\pi}z_i} \exp(-ik\tau) \quad (3.16)$$

In simple inflationary models, one usually imposes the initial conditions on the modes when $k/(aH) \approx 100$. Similarly, to compute the spectrum we need to integrate until the mode is far outside the horizon (i.e super-Hubble scales), typically, $k/aH \approx 10^{-5}$ [62, 76].

4 Methodology

4.1 CMB analysis

The initial power spectrum $\mathcal{P}(k)$ is related to the angular power spectrum C_ℓ through

$$C_\ell^{XX'} \propto \int d\ln k \mathcal{P}(k) T_\ell^X(k) T_\ell^{X'}(k), \quad (4.1)$$

where $T_\ell^X(k)$ is the transfer function with X representing the CMB temperature or polarization. The measured angular power spectrum C_ℓ is a robust cosmological probe in constraining cosmological models, the position and amplitude of the peaks being very sensitive to important cosmological parameters.

We have used modified version of CAMB [80, 81] which is based on line of sight integration approach given in [82] to calculate the angular power spectra of the CMB anisotropies for different inflationary models discussed above. For exploring the likelihood function $\mathcal{L}(\mathbf{D}|\boldsymbol{\theta}, \mathcal{M})$ (or the posterior distribution) of the parameters, we use Markov chain Monte Carlo (MCMC) methods. We have obtained the best fit values for the parameters using modified version of publicly available CosmoMC code [83] with a convergence diagnostics done through the Gelman and Rubins statistics. We have used PL model along with the PI derived scalar power spectrum for the cosmological estimates. We have conducted the analysis with two different data combinations: WMAP9 only and WMAP9+Planck. For WMAP9 data set, we have used a full temperature and polarization 9 year data and likelihood code provided by the WMAP9 team. For Planck data set, we have used low ℓ TEB likelihood ($2 \leq \ell \leq 29$) and high ℓ nuisance-marginalized Plik lite likelihood ($30 \leq \ell \leq 2500$) to compute joint likelihood for TT, EE, BB, TE.

Parameter Name	Symbol	Lower limit	Upper limit
Baryon Density	$\Omega_b h^2$	0.005	0.1
Cold Dark Matter Density	$\Omega_c h^2$	0.001	0.99
Angular size of Acoustic Horizon	θ	0.5	10.0
Optical Depth	τ	0.01	0.8
Scalar Spectral Index	n_s	0.5	1.5
Scalar Amplitude	$\ln[10^{10} A_s]$	2.7	4.0
Cut-off Parameter	a_0	0.0001	0.1
Inflection Point	ϕ_0	1.950	1.970
Inflationary Mass	$\ln[10^{10} m^2]$	-12	-8

Table 1. Uniform prior used in parameter estimation.

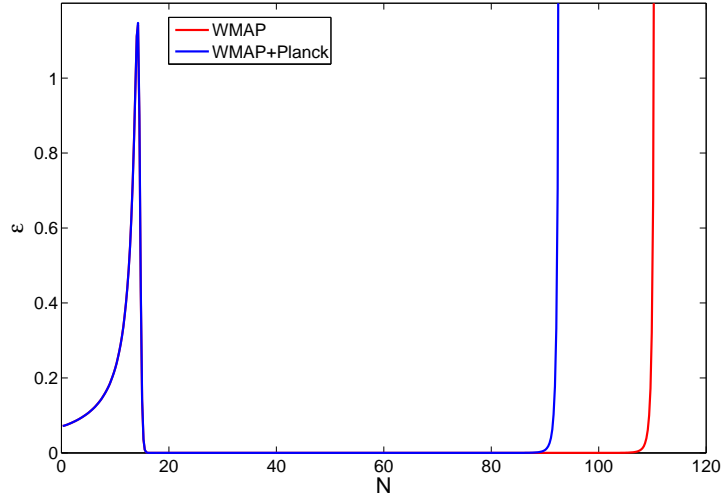


Figure 2. Evolution of the slow roll parameters ϵ as a function of e-folds N in case of PI model for WMAP9 and WMAP9+Planck data. The slow roll approximation is violated (i.e $\epsilon > 1$) for an e-fold around $N = 13 - 14$ e-folds. It can be seen the evolution of ϵ is very similar for both WMAP9 and WMAP9+Planck data sets except at the late stage of inflation. Also note that here inflation ends at relatively large value of N (i.e $N = 112$ and $N = 92$ for WMAP9 and WMAP9+Planck data sets respectively).

4.2 Model parameters and priors

The cosmological parameterization has been carried out by using the six basic parameters (baryon density “ $\Omega_b h^2$ ”, cold dark matter density “ $\Omega_c h^2$ ”, Thomson scattering optical depth due to re-ionization “ τ ”, angular size of horizon “ θ ”, “ n_s ” and scalar amplitude “ $\ln 10^{10} A_s$ ”) along with the three parameters which describe punctuated inflation (a_0 , ϕ_0 , $\ln[10^{10} m^2]$). The remaining cosmological parameters are kept constant. We have fixed the sum of physical masses of standard neutrinos “ ν ” = 0.06 eV, effective number of neutrinos “ N_{eff} ” = 3.046, Helium mass fraction “ Y_{He} ” = 0.24 and the width of re-ionization = 0.5. The results of the Bayesian parameter estimation may depend on the range of priors used. Table 1 shows the prior ranges of all parameters used in this work. Moreover, we have used flat prior distributions for all cosmological parameters.

		WMAP9			WMAP9+Planck		
Model	Parameter	Best Fit	68% Limit	$\chi^2 = -2 \log \mathcal{L}$	Best Fit	68 % Limit	$\chi^2 = -2 \log \mathcal{L}$
PL	$\Omega_b h^2$	0.0223	$0.0223^{+0.0005}_{-0.0005}$	7557.828	0.0223	$0.0213^{+0.0002}_{-0.0001}$	19206.912
	$\Omega_c h^2$	0.1141	$0.1138^{+0.0046}_{-0.0044}$		0.1225	$0.1227^{+0.0014}_{-0.0015}$	
	Ω_Λ	0.7110	$0.7124^{+0.0288}_{-0.0234}$		0.6619	$0.6611^{+0.0098}_{-0.0095}$	
	$\ln[10^{10} A_s]$	3.0869	$3.0878^{+0.0277}_{-0.0315}$		3.0801	$3.0782^{+0.0192}_{-0.0191}$	
	n_s	0.9670	$0.9683^{+0.0126}_{-0.0127}$		0.9485	$0.9483^{+0.0043}_{-0.0042}$	
	τ	0.0861	$0.0875^{+0.0127}_{-0.0147}$		0.0732	$0.0721^{+0.0096}_{-0.0096}$	
	100θ	1.0393	$1.0395^{+0.0057}_{-0.0073}$		1.0401	$1.0401^{+0.0003}_{-0.0003}$	
	z_{re}	10.551	$10.617^{+1.112}_{-1.101}$		9.8526	$9.7271^{+0.9259}_{-0.9259}$	
	H_0	68.898	$69.158^{+5.517}_{-6.563}$		65.397	$65.375^{+0.625}_{-0.626}$	
PI	$\Omega_b h^2$	0.0219	$0.0220^{+0.0005}_{-0.0003}$	7554.198	0.0213	$0.0213^{+0.0001}_{-0.0001}$	19190.034
	$\Omega_c h^2$	0.1172	$0.1164^{+0.0041}_{-0.0045}$		0.1238	$0.1234^{+0.0015}_{-0.0015}$	
	Ω_Λ	0.6914	$0.6953^{+0.0273}_{-0.0218}$		0.6536	$0.6565^{+0.0097}_{-0.0098}$	
	τ	0.0825	$0.0826^{+0.0118}_{-0.0137}$		0.0721	$0.0724^{+0.0094}_{-0.0093}$	
	100θ	1.0389	$1.0387^{+0.0020}_{-0.0020}$		1.0400	$1.0401^{+0.0003}_{-0.0003}$	
	z_{re}	10.420	$10.343^{+1.093}_{-1.083}$		9.7946	$9.7851^{+0.8999}_{-0.8716}$	
	a_0	0.0097	$0.0104^{+0.0041}_{-0.0066}$		0.0065	$0.0067^{+0.0011}_{-0.0015}$	
	ϕ_0	1.9654	$1.9661^{+0.0039}_{-0.0011}$		1.9622	$1.9626^{+0.0012}_{-0.0012}$	
	$\ln[10^{10} m^2]$	-9.8376	$-10.137^{+0.626}_{-1.196}$		-8.9618	$-9.0826^{+0.3327}_{-0.3025}$	
	H_0	67.336	$67.694^{+1.919}_{-1.738}$		64.880	$65.084^{+0.621}_{-0.620}$	

Table 2. The best fit and mean values for power law (PL) model and Punctuated Inflation (PI) model obtained using WMAP9 year and joint WMAP9+Planck data sets. We find that for the WMAP9 data set the improvement in the fit is marginal while for the WMAP9+Planck there is a significant improvement in the fit.

5 Results and discussion

5.1 Best fit parameters and joint constraints

In table 2, we present the estimates of all the parameters in terms of the best fit values and their mean values with 1- σ errors for both PL model and PI model using WMAP9 and WMAP9+Planck data. Figure 2 shows the evolution of slow roll parameter ϵ as a function of e-folds N for the best fit parameters of WMAP9 and WMAP9+Planck data set. As can be seen from figure 2 the slow roll approximation is violated for an e-fold around 13-14 e-folds. Figure 3 shows marginalized posterior distribution and two dimensional posterior distributions along with contours at 68% and 95% of the cosmological parameters for WMAP9 and WMAP9+Planck data set. Figure 4 shows best fit primordial spectra for PI model using WMAP9 and WMAP9+Planck data set. For comparison we have also plotted the best fit primordial power spectrum for the pure power law case obtained using WMAP9+Planck data set. Figures 5 & 6 show the corresponding angular power spectra C_ℓ^{TT} for the best fit values of PI model parameters and other standard cosmological parameters.

From table 2, we see that PI model gives better fit to the data than the standard power law model at the cost of one extra parameter. For WMAP9 the improvement is marginal $\Delta\chi^2 \sim 4$, however, for that of WMAP9+Planck data set there is a significant improvement

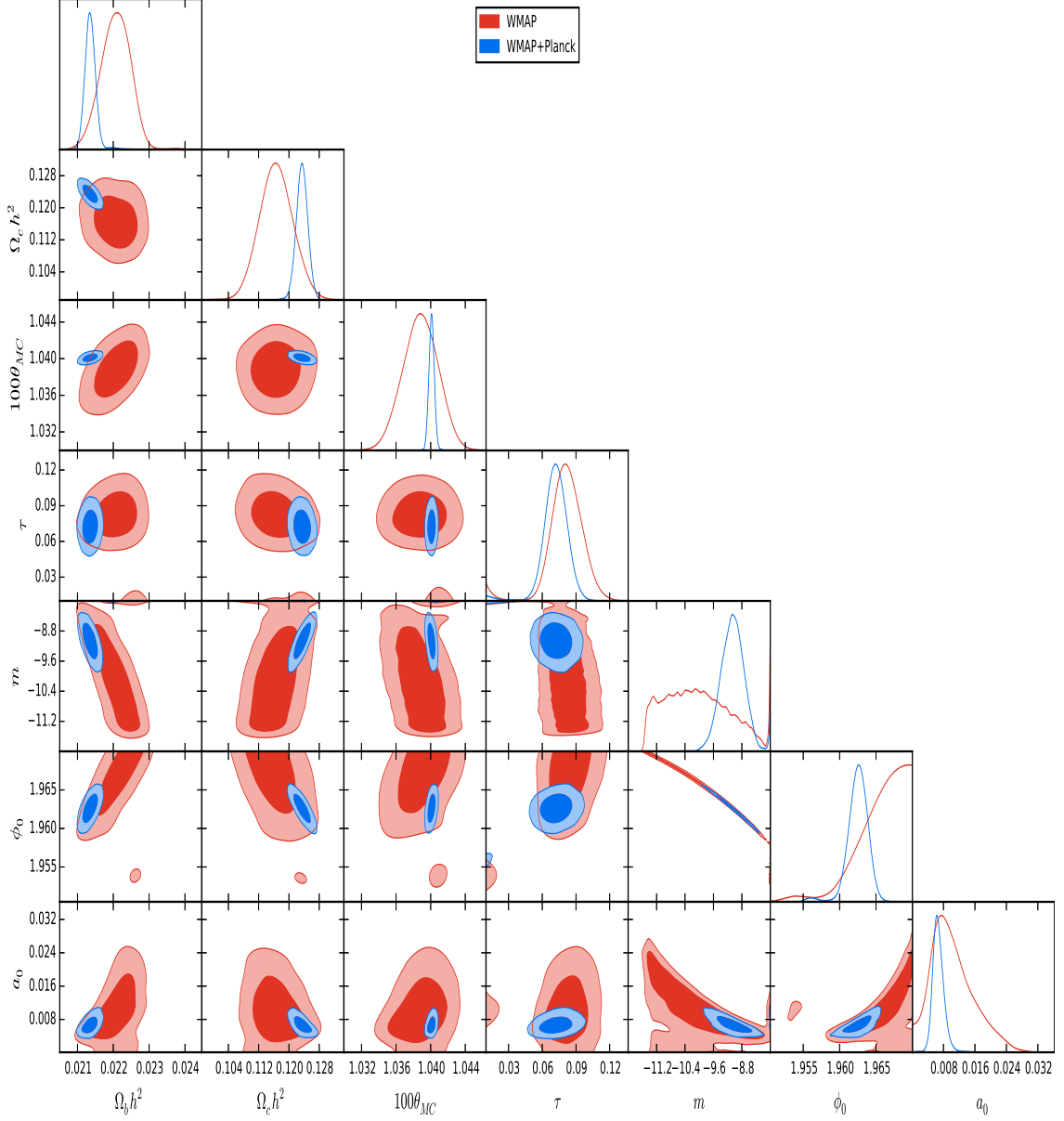


Figure 3. Two dimensional joint posterior probability distributions and one dimensional marginal posterior probability distribution for the parameters of the punctuated inflation along with other cosmological parameters.

of fit $\Delta\chi^2 \sim 17$. In the next section, we will discuss the significance of these fits using *AIC*. From figure 3, it is clear that we are able to obtain good bounds on all the parameters and that both WMAP9 and Planck data sets are consistent with each other. In fact, we were able to put much tighter constraints on the parameters with the WMAP9+Planck data set. It follows from figure 4, that the main feature of the primordial power spectrum for PI model is the existence sharp cut-off around $k \approx 4 \times 10^{-4}$ followed by a bump around $k \approx 0.001$ and suppression around $k \approx 0.002$ in both WMAP9 and WMAP9+Planck cases. This corresponds to sharp cut-off in the C_ℓ^{TT} spectrum up to multipoles $\ell \leq 4$ followed by

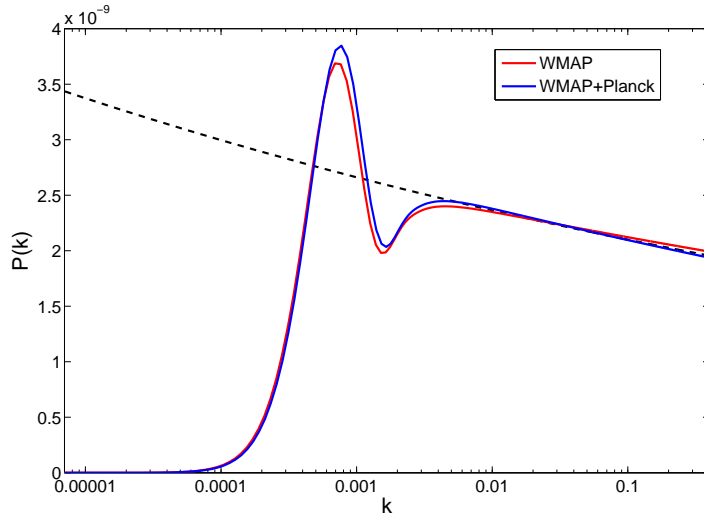


Figure 4. The best fit primordial spectra for PI model using WMAP9 (red line) and WMAP9+Planck (blue line) data sets. Note for both cases, we find sharp cut off around $k \approx 4 \times 10^{-4}$ followed by a bump and suppression. At large k , PI primordial power spectrum matches with the best fit standard power law model shown with the black dashed line for WMAP9+Planck data set.

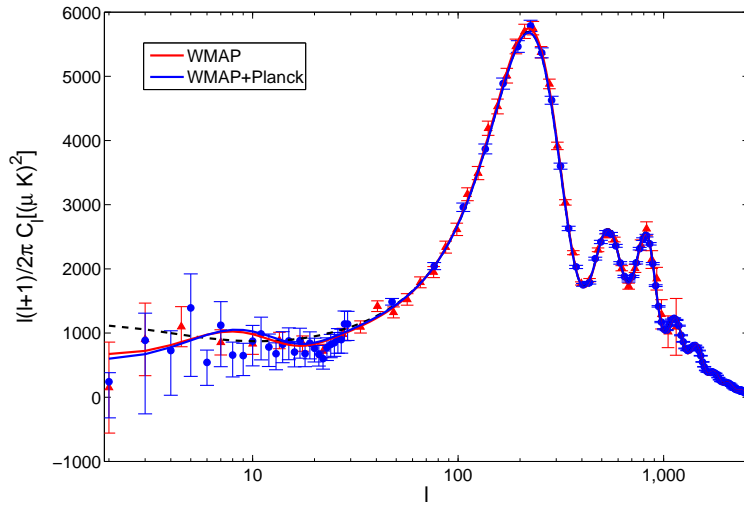


Figure 5. The best fit angular power spectra C_l^{TT} for PI model using WMAP9 and WMAP9+Planck data sets. For comparison, we have also plotted best fit C_l^{TT} using power law model for WMAP9+Planck data set with black dashed line. The observed data points for WMAP9 and Planck data are also shown by red triangles and blue dots respectively.

the bump ($5 \leq \ell \leq 11$) and suppression ($12 \leq \ell \leq 40$) as shown figure 5.

5.2 Model comparison

To judge whether a model is preferred by data, we use Akaike information criterion (*AIC*) which incorporates trade-off between the goodness of fit and additional complexity of the

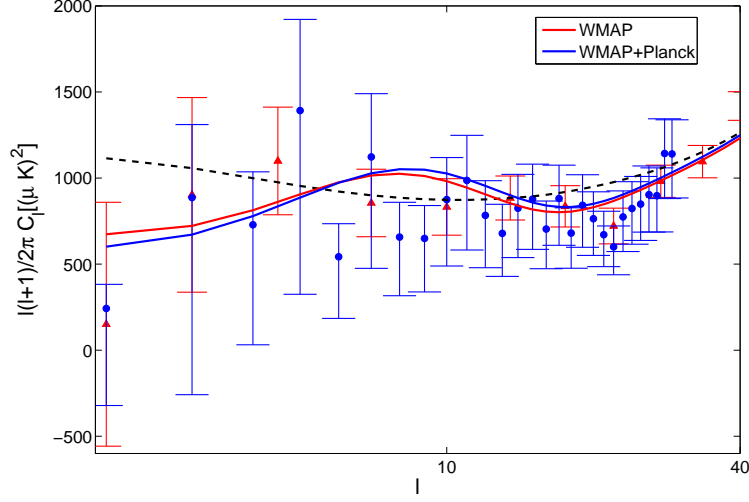


Figure 6. Same as in Fig. (5) at low- ℓ . From this figure it is clear that the PI model induces power suppression up to a large values of $l \leq 40$ with a bump around $5 \leq \ell \leq 11$ which results in the better improvement of fit in case of Planck data compared to WMAP9 data.

Data Set	Model	ΔAIC
WMAP9	PL	1.63
WMAP9+Planck	PL	14.88

Table 3. The value of ΔAIC shows the preference of PL model compared to PI model. As can be seen PI model has a substantial support compared to PL model for WMAP9+Planck data set.

model [64, 84, 85]:

$$AIC = -2 \ln \mathcal{L}_{max}(\mathbf{d}|\theta) + 2N, \quad (5.1)$$

where N is the number of free parameters and $\chi^2 = -2 \ln \mathcal{L}_{max}$, \mathcal{L}_{max} being best fit likelihood value of the given model. In the above equation, first term represents the quality of the model fit and second term represent the model complexity. The AIC is based on approximation to the Kullback-Leibler information entropy [86, 87]. The preferred model is the one which has a minimum value of AIC and $\Delta AIC \equiv AIC_i - AIC_{min}$ represent preference of model i over the best fit model (model with minimum value of AIC i.e AIC_{min}). Models with $\Delta AIC \leq 2$ have substantial support, models with $4 < \Delta AIC < 7$ have considerably less support and those with $\Delta AIC > 10$ have essentially no support compared to best fit model [88].

The values of ΔAIC given in table 3 represent the preference of PL model over the PI model. For WMAP9+Planck data set, $\Delta AIC = 14.88$ implies that PI model (which produces suppression in scalar power at small k) is preferred compared to PL model despite having an additional parameter and that PL model has essentially no support. However, for the WMAP9 only data set, both models are equally favorable, although, PI model produce slightly better likelihood.

6 Conclusions

Although power law model has emerged as the most successful model consistent with recent observations, it could not explain certain anomalies. One of these anomalies is low CMB power at large angular scales in the CMB power spectrum. Different models have been used, having their own virtues and deficiencies, to account for this power loss. For example, in our previous work [15], we studied various inflationary models using different initial condition (like kinetic or radiation dominated era). While for PI we found the change in χ^2 to be decisive, the maximum change in χ^2 was only about $\Delta\chi^2 \approx 6$ for the WMAP9+Planck data set. Similarly, there are models which superimpose oscillations on the power spectrum and which could produce $\Delta\chi^2 \approx 10 - 16$ [4, 89, 90]. However, in these cases the fitting is forced in the entire spectrum and the number of parameters involved is three or more. There also remains problem of over fitting data with such models.

Motivated by the initial condition independence, in this work, we have used the punctuated inflationary model in which a brief period of fast roll is sandwiched between two stages of slow-roll inflation. Markov Chain Monte Carlo analysis has been performed to determine posterior distributions and the values of model parameters that provide best fit to WMAP9 and Planck data for CMB angular power spectrum. We found that PI model gives better fit to the data ($\Delta\chi^2 \approx 4$ for WMAP9 and $\Delta\chi^2 \approx 17$ for WMAP9+Planck) than the standard Λ CDM model with a featureless, primordial power spectrum at the cost of one extra parameter. Further, we used Akaike information criteria for model comparison which showed that for WMAP9+Planck data, PI model (despite having an additional parameter) is preferred over PL model while as for WMAP9 data set only, both models are equally favourable with PI having slightly better likelihood. However, significant improvement, both in data and modeling, is required before one can rule out power law model.

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